Quasi-equilibrium binary black hole initial data

Harald P. Pfeiffer
California Institute of Technology

Collaborators: Greg Cook, Larry Kidder, Mark Scheel, Saul Teukolsky, James York

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Outline:
1. Formalism & Numerics
2. Non-uniqueness in conformal thin sandwich
3. Properties of the constructed ID sets
4. Public initial data repository
Formalism & Numerics
**Quasi-equilibrium method**

**Basic idea:**
Approx. *time-independence in corotating frame*
Approx. *helical Killing vector*
(both concepts essentially equivalent, both useful depending on context)

**History:**
- **Wilson & Matthews 1985:** Binary neutron stars
- **Gourgoulhon, Grandclement & Bonazzola, 2002a,b**
  BBH ID with inner boundary conditions
  basically right, but various deficiencies
  General quasi-equilibrium method with isolated horizon BCs
Quasi-equilibrium method (the easy pieces)

- Time-independence in corotating frame
  ⇒ vanishing time derivatives
Quasi-equilibrium method (the easy pieces)

- Time-independence in corotating frame
  \[ \Rightarrow \text{vanishing time derivatives} \]

- Extended conformal thin sandwich formalism

\[
\partial_t \tilde{g}_{ij} = 0 = \partial_t K
\]

\[
\tilde{\nabla}^2 \psi - \frac{1}{8} \tilde{R} \psi - \frac{1}{12} K^2 \psi^4 + \frac{1}{8} \tilde{A}_{ij} \tilde{A}^{ij} = 0
\]

\[
\tilde{\nabla}_j \left( \frac{\psi^6}{2N} \beta^{ij} \right) - \frac{2}{3} \psi^6 \tilde{\nabla}^i K - \tilde{\nabla}_j \left( \frac{\psi^6}{2N} \tilde{u}^{ij} \right) = 0
\]

\[
\tilde{\nabla}^2 (N \psi) - N \psi \left( \frac{1}{8} \tilde{R} + \frac{5}{12} K^2 \psi^4 + \frac{7}{8} \tilde{A}_{ij} \tilde{A}^{ij} \right) = -\psi^5 (\partial_t - \beta^k \partial_k) K
\]
Quasi-equilibrium method (the easy pieces)

- **Time-independence in corotating frame**
  \[ \implies \text{vanishing time derivatives} \]

- **Extended conformal thin sandwich formalism**
  \[ \partial_t \tilde{g}_{ij} = 0 = \partial_t K \]

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- **Boundary conditions at infinity**
  \[ \psi = 1 \]
  \[ \beta^i = (\vec{\Omega}_{\text{orbital}} \times \vec{r})^i \]
  \[ N = 1 \]

- **New contribution:** *inner boundary conditions* (next slides)
Quasi-equilibrium excision boundary conditions

- **Excise** topological spheres $S$

- **Require**
  1. $S$ be apparent horizons
  2. The AH’s remain stationary in evolution
  3. Shear of $k^{\mu}$ vanishes (isolated horizon)

\[ \Rightarrow \mathcal{L}_k \theta = 0 \Rightarrow \text{AH moves along } k^{\mu} \text{ and } M_{AH} \text{ initially constant} \]
Quasi-equilibrium excision boundary conditions

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- **Rewrite** in conformal variables \( \Rightarrow \) BC’s on \( \psi \) and \( \beta^i \)
Quasi-equilibrium excision boundary conditions

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- **Rewrite** in conformal variables \( \Rightarrow \) BC’s on \( \psi \) and \( \beta^i \)

- General spin possible (\( \rightarrow \) Greg Cook’s talk)
Quasi-equilibrium excision boundary conditions

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- **Require**
  1. $S$ be apparent horizons
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  $\Rightarrow \mathcal{L}_k \theta = 0 \Rightarrow \text{AH moves along } k^\mu \text{ and } M_{\text{AH}} \text{ initially constant}$

- **Rewrite** in conformal variables $\Rightarrow$ BC’s on $\psi$ and $\beta^i$

- General spin possible (→ Greg Cook’s talk)

- One still must specify...
  1. Conformal metric $\tilde{g}_{ij}$
  2. Shape of excision surfaces $S$
  3. Mean curvature $K$
  4. Lapse boundary condition
Spectral elliptic solver  

(HP, Kidder, Scheel & Teukolsky, 2003)

Expand solution in basis-functions & solve for expansion-coefficients
**Spectral elliptic solver** *(HP, Kidder, Scheel & Teukolsky, 2003)*

Expand solution in basis-functions & solve for expansion-coefficients

Smooth solutions $\Rightarrow$ exponential convergence
Spectral elliptic solver  
(HP, Kidder, Scheel & Teukolsky, 2003)

Expand solution in basis-functions & solve for expansion-coefficients

Smooth solutions ⇒ exponential convergence

- Superior accuracy: Numerical errors ≪ physical effects
Spectral elliptic solver

(HP, Kidder, Scheel & Teukolsky, 2003)

Expand solution in basis-functions & solve for expansion-coefficients

Smooth solutions $\Rightarrow$ exponential convergence

- **Superior accuracy:** Numerical errors $\ll$ physical effects
- **Superior efficiency:** Large parameter studies

\[ ||H||_2 \quad ||M||_2 \quad |dE_{ADM}| \quad |dM_k| \quad |dJ_z| \]

HP, Kidder, Scheel, Teukolsky 2003
Spectral elliptic solver

(HP, Kidder, Scheel & Teukolsky, 2003)

Expand solution in basis-functions & solve for expansion-coefficients

Smooth solutions $\Rightarrow$ exponential convergence

- **Superior accuracy:** Numerical errors $\ll$ physical effects
- **Superior efficiency:** Large parameter studies
- **Domain decomposition:** Nontrivial topologies & Multiple length-scales

HP, Kidder, Scheel, Teukolsky 2003
Non-uniqueness
Extended conformal thin sandwich equations

\[ \tilde{g}_{ij} = \delta_{ij} + \tilde{A}_h_{ij} \]
\[ \partial_t \tilde{g}_{ij} = \tilde{A}_h_{ij} \]
\[ K = \partial_t K = 0 \]
(perturbed flat space w/o inner b’dries)

ADM energy

HP & York, 2005

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Extended conformal thin sandwich equations

\[ \tilde{g}_{ij} = \delta_{ij} + \tilde{A}h_{ij} \]

\[ \partial_t \tilde{g}_{ij} = \tilde{A}\dot{h}_{ij} \]

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HP & York, 2005
Extended conformal thin sandwich equations

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\[ K = \partial_t K = 0 \]

(perturbed flat space w/o inner b’dries)

Apparent horizons exist for small \( \tilde{A} \)!
Properties of QE-ID sets
Corotating BBH solutions

**Arbitrary choices:** Conformal flatness, $S = \text{sphere}$. Gauge choices: $K = 0$, $\partial_n(N\psi) = 0$.

**Exponential convergence**

**Laplace positive through horizon**
Sequences of quasi-circular orbits & ISCO

\[ \frac{E_b}{\mu} \sim \frac{J}{\mu m} \]

-0.06
-0.05
-0.04
-0.03
-0.02

\[ \frac{E_b}{m} \sim m \Omega_0 \]

0.08 0.12 0.16

1,2,3 PN (EOB)
2,3 PN (standard)

Cook&Pfeiffer '04
(3 data points)

GGB '02

Cook&Pfeiffer '04
(3 data points)

PN (EOB)
PN (standard)

Cook '94

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Towards evolving these ID

- ISCO and other diagnostics very promising

- But, ID only **up to** AH, whereas evolution codes excise **inside** AH

- Extrapolate data inward to $0.75r_{AH}$

- Constraints violated for $r < r_{AH}$

- The next slides highlight aspects of evolution which are relevant to ID

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Evolution with fixed gauge – horizon motion

Same data as in Mark Scheel’s talk – separation 10.

Initially at rest, no transient

$N$ and $\beta^i$ are excellent initial gauge
Evolution with fixed gauge – horizon motion

Same data as in Mark Scheel’s talk – separation 10.

Initially at rest, no transient

On longer time-scales, AH deforms

$N$ and $\beta^i$ are excellent initial gauge
Apparent horizon mass

\[ \frac{M_{AH}}{M_{AH}(t=0)} \]

(3 different resolutions)

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Tidal distortions not captured correctly with current choices for $\tilde{g}_{ij}$ and $S$

--- Work in progress ---
Public ID repository
Initial data repository

- [http://www.tapir.caltech.edu/~harald/PublicID](http://www.tapir.caltech.edu/~harald/PublicID)
- Equal mass BBHs in corotation
- Two choices for Lapse-BC – Eq. (59a) or (59b) from Cook&HP, 2004
Initial data repository

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Concentrate on Lapse-BC (59a) for uniformity
Initial data repository

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Concentrate on Lapse-BC (59a) for uniformity

Available separations

- Pretorius
- Bruegmann et al 2004

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Using the public QE-BBH initial data

http://www.tapir.caltech.edu/~harald/PublicID

The web-site contains:

- Data sets, containing $g_{ij}$, $K_{ij}$, $N$, $\beta^i$ in **Cartesian** components

- Library to interpolate the data to any desired point $(x, y, z)$
  (as long as it is inside the covered computational domain)

- Example executable and example data-set
  (Schwarzschild in Kerr-Schild coordinates)
Summary

- Framework for BBH initial data in a kinematical setting (helical Killing vector)

- Advantages:
  1. Agreement with PN
  2. $N > 0$, AH initially constant, $M_{\text{AH}}$ exceedingly constant

- Tidal distortions not yet captured

- Data sets publicly available
  
  http://www.tapir.caltech.edu/~harald/PublicID

  1. Compute waveforms!
  2. Compare and validate evolution codes on the same initial data
Contents of a data set

1. **The data** in several resolutions (Lev2, ... Lev5), each in its own subdirectory

2. The file **Convergence** listing errors for each resolution:

<table>
<thead>
<tr>
<th>#....N</th>
<th>Nor-Linf</th>
<th>Nor-L2</th>
<th>Ham-Linf</th>
<th>Ham-L2</th>
<th>Mom-Linf</th>
<th>Mom-L2</th>
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</thead>
<tbody>
<tr>
<td>32.184</td>
<td>0.2280</td>
<td>0.03185</td>
<td>0.0339</td>
<td>0.00202</td>
<td>0.00052</td>
<td>0.000217</td>
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<tr>
<td>46.447</td>
<td>0.0001337</td>
<td>2.452e-05</td>
<td>0.00333</td>
<td>0.000119</td>
<td>0.000461</td>
<td>1.03e-05</td>
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<tr>
<td>60.706</td>
<td>4.361e-06</td>
<td>9.697e-07</td>
<td>0.000238</td>
<td>6.12e-06</td>
<td>2.80e-05</td>
<td>4.14e-07</td>
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<tr>
<td>74.963</td>
<td>1.432e-07</td>
<td>3.253e-08</td>
<td>1.40e-05</td>
<td>2.71e-07</td>
<td>1.48e-06</td>
<td>1.62e-08</td>
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<tr>
<td>89.219</td>
<td>4.855e-09</td>
<td>6.189e-10</td>
<td>7.38e-06</td>
<td>1.13e-08</td>
<td>6.98e-08</td>
<td>6.11e-10</td>
</tr>
<tr>
<td>103.47</td>
<td>3.065e-10</td>
<td>1.267e-11</td>
<td>3.59e-08</td>
<td>4.45e-10</td>
<td>3.08e-09</td>
<td>2.24e-11</td>
</tr>
</tbody>
</table>

   Change between this and Hamiltonian and momentum constraints
   next lower resolution outside horizon

3. The file **AdmQuantities** listing some relevant quantities at each resolution

<table>
<thead>
<tr>
<th></th>
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<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>32.184</td>
<td>4.4323659273442</td>
<td>1.14349189174678</td>
<td>2.250487955424110</td>
</tr>
<tr>
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<td>1.14359857991925</td>
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<td>89.219</td>
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</tr>
<tr>
<td>103.47</td>
<td>4.4405875503605</td>
<td>1.14360454285761</td>
<td>2.250630340509140</td>
</tr>
</tbody>
</table>

4. The file **Omega** containing the orbital angular frequency
Interpolation Library – suggestions welcome!

- **Library** libSpECLibraryID.a (compiled with gcc 3.4.3 on RHE 9)

- **Header file** PublicID.hpp:

  ```cpp
  #include <vector>

  void ReadData(const double Omega);  // import from disk

  void InterpolateData(const std::vector<double>& x,
                        const std::vector<double>& y,
                        const std::vector<double>& z,
                        std::vector<double>& gxx, ... , std::vector<double>& gzz,
                        std::vector<double>& Kxx, ... , std::vector<double>& Kzz,
                        std::vector<double>& Betax, ... , std::vector<double>& Betaz,
                        std::vector<double>& N);

  void ReleaseData();  // free memory
  
  - **Test-executable** InterpolateExample.cpp:

    ```cpp
    g++ InterpolateExample.cpp libSpECLibraryID.a -lblas
    ```